MASx50: Assignment 2

1. The following text describes the key steps of defining the Lebesgue integral on a measure space (S, Σ, m). It contains *three* mistakes.

1 For indicator functions $\mathbb{1}_A$ where $A \in \Sigma$, set

2 $$\int_0^\infty \mathbb{1}_A \, dm = m(A). \qquad (\star)$$

3 For simple functions $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$, where $c_i \in \mathbb{R}$ and $A_i \in \Sigma$ for all $i \in$

4 $\{1, \ldots, n\}$, extend equation $(\star)$ by linearity to give

5 $$\int_S s \, dm = \sum_{i=1}^n c_i m(A_i).$$

6 For non-negative measurable functions $f : S \to [0, \infty)$, define

7 $$\int_S f \, dm = \sup \left\{ \int_S s \, dm \ : \ s \text{ is a continuous function and } 0 \le s \le f \right\}.$$

8 We therefore have that $\int_S f \, dm \in [0, \infty)$ for non-negative measurable functions f.

9 For an arbitrary measurable function $f : S \to \mathbb{R}$, write $f = f_+ - f_-$, where

10 $f_+ = 0 \vee f$ and $f_- = -(f \wedge 0)$. Then f_+ and f_- are non-negative measurable

11 functions. If one or both of $\int_S f_+ \, dm$ and $\int_S f_- \, dm$ is not equal to $+\infty$ then we

12 define

13 $$\int_S f \, dm = \int_S f_+ \, dm - \int_S f_- \, dm.$$

14 If both $\int_S f_+ \, dm$ and $\int_S f_- \, dm$ are equal to $+\infty$ then $\int_S f \, dm$ is undefined.

Each mistake is on a distinct line. Line numbers are included for convenience and to help you reference the text.

List the line numbers containing mistakes and, for each mistake, give a corrected version.

2. Determine if the following functions are in $\mathcal{L}^1$. Use the monotone convergence theorem to justify your answers.

   (a) $f : (1, \infty) \to \mathbb{R}$ by $f(x) = 1/x^2$.

   (b) $g : (-1, 1) \to \mathbb{R}$ by $g(x) = 1/x^3$, where we set $g(0) = 0$.

3. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

   (a) Let $f : S \to \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.

(b) Let $f : S \to \mathbb{R}$ be a simple function. Write $f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$ where the A_i are pairwise disjoint and measurable and $c_i \geq 0$. Show that

$$\left(\int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \tag{$\star$}$$

Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

$$\frac{\left(\sum_{i=1}^{n} u_i\right)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}.$$

(c) In this question you should give *two* different proofs that equation ($\star$) holds when f is any non-negative measurable function. You may use your results from part (b) in both proofs.

i. Give a proof using the monotone convergence theorem.

ii. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.

(d) Does ($\star$) remain true if m is not necessarily a probability measure?