# MASx50: Assignment 2

Solutions and discussion are written in blue. Some common pitfalls are indicated in teal. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

Marks are given for [A]*ccuracy,* [J]*ustification, and* [M]*ethod.*

1. The following text describes the key steps of defining the Lebesgue integral on a measure space $(S, \Sigma, m)$. It contains *three* mistakes.

1 For indicator functions $\mathbb{1}_A$ where $A \in \Sigma$, set

2 $$~~\int_0^\infty~~ \int_S \mathbb{1}_A \, dm = m(A). \qquad (\star)$$

3 For simple functions $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$, where $c_i \in \mathbb{R}$ and $A_i \in \Sigma$ for all
4 $i \in \{1, \ldots, n\}$, extend equation $(\star)$ by linearity to give

5 $$\int_S s \, dm = \sum_{i=1}^n c_i m(A_i).$$

6 For non-negative measurable functions $f : S \to [0, \infty)$, define

7 $$\int_S f \, dm = \sup \left\{ \int_S s \, dm \; : \; s \text{ is a ~~continuous~~ simple function and } 0 \le s \le f \right\}.$$

8 We therefore have that $\int_S f \, dm \in$ ~~$[0, \infty)$~~ $[0, \infty]$ for non-negative measurable functions $f$.

9 For an arbitrary measurable function $f : S \to \mathbb{R}$, write $f = f_+ - f_-$, where
10 $f_+ = 0 \vee f$ and $f_- = -(f \wedge 0)$. Then $f_+$ and $f_-$ are non-negative measurable
11 functions. If one or both of $\int_S f_+ \, dm$ and $\int_S f_- \, dm$ is not equal to $+\infty$ then
12 we define

13 $$\int_S f \, dm = \int_S f_+ \, dm - \int_S f_- \, dm.$$

14 If both $\int_S f_+ \, dm$ and $\int_S f_- \, dm$ are equal to $+\infty$ then $\int_S f \, dm$ is undefined.

Each mistake is on a distinct line. Line numbers are included for convenience and to help you reference the text.

List the line numbers containing mistakes and, for each mistake, give a corrected version.

*Solution.*

(a) 2, 7, 8. [3A]

(b) As indicated above. [3A]

2. Determine if the following functions are in $\mathcal{L}^1$. Use the monotone convergence theorem to justify your answers.

   (a) $f : (1, \infty) \to \mathbb{R}$ by $f(x) = 1/x^2$.

   (b) $g : (-1, 1) \to \mathbb{R}$ by $g(x) = 1/x^3$, where we set $g(0) = 0$.

*Solution.*

(a) Note that $x^{-2} > 0$ for $x \in (1, \infty)$. By Riemann integration, we have

$$\int_1^n x^{-2}\,dx = \left[-x^{-1}\right]_1^n = -\frac{1}{n} + 1. \tag{†}$$

[1A] Note that $f_n(x) = x^{-2}\mathbb{1}_{(1,n)}(x)$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $f(x) = x^{-2}$ for $x \in (1, \infty)$. [1J] Hence, by the monotone convergence theorem [1M] we have

$$\int_1^\infty x^{-2}\,dx = \lim_{n\to\infty}\left(-\frac{1}{n} + 1\right) = 1.$$

Thus $f(x) = x^{-2}$ is in $\mathcal{L}^1$ on $(1, \infty)$. [1A]

(b) The function $g$ is discontinuous at $x = 0$ (sketch it!), with $g(x) < 0$ for $x < 0$ and $g(x) > 0$ for $x > 0$. Hence $g_+(x) = \mathbb{1}_{(0,1)}x^{-3}$ and $g_-(x) = -\mathbb{1}_{(0,1)}x^{-3}$.

We will show that $\int_0^1 g_+(x)\,dx = \infty$, which means that $g \notin \mathcal{L}^1$ on $(0, 1)$. [1M]

By Riemann integration, we have

$$\int_{1/n}^1 x^{-3}\,dx = \left[\frac{x^{-2}}{-2}\right]_{1/n}^1 = \frac{1}{-2} - \frac{(1/n)^{-2}}{-2} = \frac{n^2}{2} - \frac{1}{2}.$$

[1A] We have that $g_n(x) = x^{-3}\mathbb{1}_{x\in(1/n,1)}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $g(x) = x^{-3}$ for $x \in (0, 1)$. [1J] Hence, by the monotone convergence theorem, $\int_0^1 g_+(x)\,dx = \lim_{n\to\infty}\left(\frac{n^2}{2} - \frac{1}{2}\right) = \infty$. [1A]

*Pitfall:* In order to apply Riemann integration we need to have a continuous function on a closed bounded interval. For this reason in (a) we need to avoid the limit $x = +\infty$ (because that would give an unbounded interval of $x$) when calculating (†). In (b) we need to avoid $x = 0$ because $g(x) \to \infty$ as $x \searrow 0$ and $g(x) \to -\infty$ as $x \nearrow 0$.

If we try to ignore this restriction then we can run into trouble. For example in (b) we might end up writing $\int_{-1}^1 x^{-3}\,dx = [\frac{x^{-2}}{-2}]_{-1}^1 = \frac{1}{-2} - \frac{1}{-2} = 0$, which isn't true. According to the definition of the Lebesgue integral $\int_{-1}^1 x^{-3}\,dx$ is undefined, because both $\int_{-1}^1 g_+(x)\,dx$ and $\int_{-1}^1 g_-(x)\,dx$ are infinite; the equation $\int_{-1}^1 g(x)\,dx = \int_{-1}^1 g_+(x)\,dx - \int_{-1}^1 g_-(x)\,dx = \infty - \infty$ is nonsense.

3. Let $(S, \Sigma, m)$ be a measure space, and suppose that $m$ is a probability measure.

   (a) Let $f : S \to \mathbb{R}$ be a non-negative simple function. Show that $f^2$ is also a non-negative simple function.

   (b) Let $f : S \to \mathbb{R}$ be a simple function. Write $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ where the $A_i$ are pairwise disjoint and measurable and $c_i \geq 0$. Show that

$$\left( \int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \tag{$\star$}$$

   *Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,*

$$\frac{\left(\sum_{i=1}^n u_i\right)^2}{\sum_{i=1}^n v_i} \leq \sum_{i=1}^n \frac{u_i^2}{v_i}.$$

   (c) In this question you should give *two* different proofs that equation ($\star$) holds when $f$ is any non-negative measurable function. You may use your results from part (b) in both proofs.

      i. Give a proof using the monotone convergence theorem.

      ii. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.

   (d) Does ($\star$) remain true if $m$ is not necessarily a probability measure?

*Solution.*

(a) We have

$$f^2 = \sum_{i=1}^n \sum_{j=1}^m c_i c_j \mathbb{1}_{A_i} \mathbb{1}_{A_j} = \sum_{i=1}^n c_i^2 \mathbb{1}_{A_i}$$

where the second inequality follows by disjointness – all the cross terms (when $i \neq j$) are zero. [1A] We have thus expressed $f^2$ as a simple function, and since $c_i^2$ are non-negative, $f^2$ is also non-negative. [1J]

(b) We have

$$\left( \int f \, dm \right)^2 = \left( \sum_{i=1}^n c_i m(A_i) \right)^2,$$

$$\int f^2 \, dm = \sum_{i=1}^n c_i^2 m(A_i).$$

[2A] The required inequality follows from the above and Titu's lemma, taking $v_i = m(A_i)$ and $u_i = c_i m(A_i)$. [1A] Note that, because $m$ is a probability measure, $\sum_i m(A_i) = 1$ and we may assume $m(A_i) > 0$ (because any $A_i$ with zero measure will have no effect on the value of the integral).

*Follow-up challenge exercise: See if you can derive Titu's lemma from the Cauchy-Schwarz inequality.*

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be non-negative and measurable.

**First proof (using the monotone convergence theorem):** From lectures (see the section on simple functions) there exists a sequence $(s_n)$ of non-negative simple functions such that $0 \le s_n \le s_{n+1} \le f$ such that $s_n \to f$ pointwise. [1M] Thus, by the monotone convergence theorem, as $n \to \infty$,

$$\int s_n \, dm \to \int f \, dm.$$

[1M] By part (a), $(s_n^2)$ is also a sequence of simple functions. [1J] We have $0 \le s_n^2 \le s_{n+1}^2 \le f^2$, also $s_n^2 \to f^2$ pointwise. So by another application of the monotone convergence theorem we have

$$\int s_n^2 \, dm \to \int f^2 \, dm.$$

[1M] From part (b) we have

$$\left( \int s_n \, dm \right)^2 \le \int s_n^2 \, dm$$

for all $n$. Since limits preserve weak inequalities, [1J] we have that

$$\left( \int f \, dm \right)^2 \le \int f^2 \, dm$$

as required.

**Second proof (using the definition of the integral):** Recall that the definition of the Lebesgue integral, for non-negative measurable functions, is

$$\int f \, dm = \sup \left\{ \int s \, dm \;:\; s \text{ is simple and } 0 \le s \le f \right\}.$$

Hence

$$\begin{aligned}
\left( \int f \, dm \right)^2 &= \left( \sup \left\{ \int s \, dm \;:\; s \text{ is simple and } 0 \le s \le f \right\} \right)^2 \\
&= \sup \left\{ \left( \int s \, dm \right)^2 \;:\; s \text{ is simple and } 0 \le s \le f \right\} \\
&\le \sup \left\{ \int s^2 \, dm \;:\; s \text{ is simple and } 0 \le s \le f \right\} \\
&= \sup \left\{ \int r \, dm \;:\; r \text{ is simple and } 0 \le r \le f^2 \right\} \\
&= \int f^2 \, dm
\end{aligned}$$

[1M] Here, the second line follows because $\int s \, dm \ge 0$, so the square can pass inside of the sup. [1J] The third line then follows by part (b). [J] Let us now justify the fourth line. We have shown in (a) that if $s$ is a non-negative simple function then so is $r = s^2$, and clearly if $s \le f$ then $s^2 \le f^2$ (i.e. pointwise). [1J] Also, if $r$ is a non-negative simple function such that $0 \le r \le f^2$, then if we define $s = \sqrt{r}$, we can show (in similar style to part (a)) that $s$ is a non-negative simple function such that $0 \le s \le f$. Here, if $r = \sum_i c_i \mathbb{1}_{A_i}$ we would have $s = \sum_i \sqrt{c_i} \mathbb{1}_{A_i}$. So, the two sups in the third and fourth lines are equal using the correspondence $r = s^2$. [1J]

(d) In general ($\star$) fails when $m$ is not a probability measure – almost any example you check will show that it fails.
For example, take $f(x) = x$ and let $m$ be Lebesgue measure on $[0, 2]$. Then $\int_0^2 x\,dx = 2$ and $\int_0^2 x^2\,dx = \frac{8}{3}$, but $2^2 > \frac{8}{3}$. [2A]

Total marks: 30