MASx50: Assignment 3

1. Let $f_n, f : [0,1] \to \mathbb{R}$. In each of the following cases, explain whether the Monotone and/or Dominated Convergence Theorems can be used to prove that $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$.

   (a) $f_n(x) = \cos(\frac{x}{n}) + \sin(\frac{x}{n})$ and $f(x) = 1$.

   (b) $f_n(x) = \mathbb{1}_{[\frac{1}{n},1]}(x)\,x^{-1}$ and $f(x) = \mathbb{1}_{(0,1]}x^{-1}$.

   (c) $f_n(x) = \mathbb{1}_{[0,\frac{1}{n}]}(x)\,n$ and $f(x) = 0$.

2. Let (S, Σ, m) be a measure space. Let $f : S \to \mathbb{R}$ be measurable and let $c > 0$. Consider the following two facts, which were stated (and proved) within the lecture notes:

   (a) $\displaystyle \left|\int_S f\,dm\right| \leq \int_S |f|\,dm,$

   (b) $\displaystyle m(\{x \in S \,:\, |f(x)| \geq c\}) \leq \frac{1}{c}\int_S |f|\,dm.$

   You do *not* need to prove these facts here.

   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable. Let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Use this notation to write down probabilistic versions of statements (a) and (b).

3. Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ denotes the restriction of Lebesgue measure to the Borel σ-field $\mathcal{B}([0,1])$ on $[0,1]$.

   Let $X_n(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ \omega n^{3/2} & \text{if } \omega \in (0, \frac{1}{n}] \\ 0 & \text{if } \omega \in (\frac{1}{n}, 1]. \end{cases}$

   Determine in which modes of convergence we have $X_n \to 0$.

4. (a) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0,1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$.

   (b) Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.