MASx50: Assignment 3

Solutions and discussion are written in blue. Some common pitfalls are indicated in teal. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

Marks are given for [A] ccuracy, [J] ustification, and [M] ethod.

- 1. Let $f_n, f: [0,1] \to \mathbb{R}$. In each of the following cases, explain whether the Monotone and/or Dominated Convergence Theorems can be used to prove that $\int_0^1 f_n(x) dx \to \int_0^1 f(x) dx$.
 - (a) $f_n(x) = \cos(\frac{x}{n}) + \sin(\frac{x}{n})$ and f(x) = 1.
 - (b) $f_n(x) = \mathbb{1}_{[\frac{1}{n},1]}(x) x^{-1}$ and $f(x) = \mathbb{1}_{(0,1]}x^{-1}$.
 - (c) $f_n(x) = \mathbb{1}_{[0,\frac{1}{n}]}(x) n$ and f(x) = 0.

Solution.

- (a) DCT only (the MCT can't be used here because $f_n \leq f_{n+1}$ doesn't hold). [1A + 1J]
- (b) MCT only (the DCT can't be used here because $\int_0^1 f(x) dx = \infty$). [1A + 1J]
- (c) Neither, in this case $\int_0^1 f_n(x) dx = 1$ and $\int_0^1 f(x) dx = 0$. [1A + 1J]
- 2. Let (S, Σ, m) be a measure space. Let $f : S \to \mathbb{R}$ be measurable and let c > 0. Consider the following two facts, which were stated (and proved) within the lecture notes:

(a)
$$\left| \int_{S} f \, dm \right| \leq \int_{S} |f| \, dm,$$

(b) $m(\{x \in S : |f(x)| \geq c\}) \leq \frac{1}{c} \int_{S} |f| \, dm.$

You do *not* need to prove these facts here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable. Let \mathbb{E} denote expectation with respect to \mathbb{P} . Use this notation to write down probabilistic versions of statements (a) and (b).

Solution.

- (a) $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$. [2A]
- (b) $\mathbb{P}[|X| \ge c] \le \frac{1}{c} \mathbb{E}[|X|]$. [2A]

3. Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ where λ denotes the restriction of Lebesgue measure to the Borel σ -field $\mathcal{B}([0, 1])$ on [0, 1].

Let
$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega = 0\\ \omega n^{3/2} & \text{if } \omega \in (0, \frac{1}{n}]\\ 0 & \text{if } \omega \in (\frac{1}{n}, 1]. \end{cases}$$

Determine in which modes of convergence we have $X_n \to 0$ (i.e. almost sure, in probability, in L^p and in distribution).

Solution.

We first check almost sure convergence. Fix some $\omega \in [0,1]$. If $\omega \in (0,1]$ then for all large enough *n* we have $\frac{1}{n} < \omega$. For such *n* we have $X_n(\omega) = 0$, [1M] which means $X_n(\omega) \to 0$. We thus obtain that $\{X_n(\omega) \to 0\} \subseteq (0,1]$ [1J] so

$$\lambda(\{X_n(\omega) \to 0\}) \ge \lambda((0,1]) = 1,$$

which means that $X_n \stackrel{a.s.}{\to} 0$. [1A]

It follows that $X_n \xrightarrow{\mathbb{P}} 0$ and also that $X_n \xrightarrow{d} 0$. [1A + 1J]

Lastly, the expectation of $|X_n|^p = X_n^p$ is given by

$$\mathbb{E}[X_n^p] = \int_0^1 X_n(\omega)^p \, d\lambda(\omega)$$
$$= \int_0^{\frac{1}{n}} \omega^p n^{3p/2} \, d\lambda(\omega)$$
$$= n^{3p/2} \left[\frac{\omega^{p+1}}{p+1}\right]_0^{\frac{1}{n}}$$
$$= n^{3p/2} \frac{(1/n)^{p+1}}{p+2}$$
$$= n^{p/2-1}.$$

[1M] Here we use that $\{0\}$ is a λ -null subset of [0, 1] (so values of X_n here have no effect on the integral) [1J] and that $X_n(\omega) = 0$ when $\omega > \frac{1}{n}$. [1J]

Noting that $n^{p/2-1} \to 0$ if and only if p < 2, we have that $X_n \stackrel{L^p}{\to} 0$ if and only if p < 2. [1A] For the last part, I would accept "p = 1 works but p = 2, 3, 4... does not"

- 4. (a) Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on (0, 1). Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$.
 - (b) Let $(X_n)_{n \in \mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.

Solution.

(a) We have $\mathbb{P}[U_n \leq a] = a$. For any (deterministic) sequence (x_n) the events $\{U_n < x_n\}$ are independent, because the U_n are independent. [1J] Noting that $\sum 1/n = \infty$ and $\sum 1/n^2 < \infty$, we have $\sum_n \mathbb{P}[U_n < 1/n] = \infty$ and $\sum_n \mathbb{P}[U_n < 1/n^2] < \infty$. [1J] By the second Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and by the first Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$. [1J]

(b) Let $E_n = \{X_n + i = i \text{ for } i = 1, 2, 3, 4, 5, 6\}$. We have $\mathbb{P}[E_n] = (1/6)^6 > 0$. Note that E_n and E_{n+6} are independent (but E_n and E_{n+1} are not!). [1M] We have $\sum_{n=1}^{\infty} \mathbb{P}[E_{6n}] = \sum_{n=1}^{\infty} (1/6)^n = \infty$, [1A] hence by the second Borel-Cantelli lemma we have $\mathbb{P}[E_{6n} \text{ i.o.}] = 1$. [1J] Noting that $\{E_{6n} \text{ i.o.}\} \subseteq \{E_n \text{ i.o.}\}$, we have $\mathbb{P}[E_n \text{ i.o.}] = 1$.

Total marks: 25