MASx52: Assignment 1

1. Recall the one-period market, and its parameters r, u, d, p_u, p_d and s. We assume that $d < 1 + r < u$.

   (a) At time $t = 0$ our portfolio contains 2 unit of cash and 3 units of stock. What is the value of our portfolio at time $t = 0$? If we hold this portfolio until time $t = 1$, what is its new value?

   (b) A rival investor holds a portfolio containing 3 units of cash and 2 unit of stock. Under what condition (on the parameters) can we be *certain* that our own portfolio will have a strictly greater value at time $t = 1$?

2. Let $\Omega = \{HH, HT, TH, TT\}$, representing two coin tosses each of which may show either H (head) or T (tail). Let $X : \Omega \to \mathbb{R}$ be the toss in which the first head occurred, or zero if no heads occurred:

$$X = \begin{cases} 0 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT \text{ or } \omega = HH \\ 2 & \text{if } \omega = TH. \end{cases}$$

   Let Y be the total number of heads that occurred in both tosses.

   (a) Write down the sets $X^{-1}(0)$, $X^{-1}(1)$ and $X^{-1}(2)$.

   (b) List the elements of $\sigma(X)$.

   (c) Is Y measurable with respect to $\sigma(X)$? Justify your answer.

3. Let $\Omega = \{1, 2, 3, 4, 5\}$, representing one roll of a five sided dice. In each case, match the σ-field to the information it contains.

   (a) $\big\{\emptyset, \Omega, \{1\}, \{2, 3, 4, 5\}\big\}$

   (b) $\sigma(\{1, 2, 3\}, \{3, 4, 5\})$

   (c) $\big\{\emptyset, \Omega, \{1\}, \{2, 3, 4\}, \{5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 5\}\big\}$

   (i) If the roll was less than or equal to 3.

   (ii) If the roll was the minimum possible value, or the maximum possible value, or neither.

   (iii) If the roll was equal to 3, or strictly less three, or strictly greater than 3.

   (iv) If the roll was a 1 or not.

4. Let X be a random variable.

   (a) Show that $Y = \cos X$ is also a random variable.

   (b) For which $p \in [1, \infty)$ do we have $Y \in L^p$?