MASx52: Assignment 3

Solutions and discussion are written in blue. Some common pitfalls are indicated in teal. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

Marks are given for [A]*ccuracy,* [J]*ustification, and* [M]*ethod.*

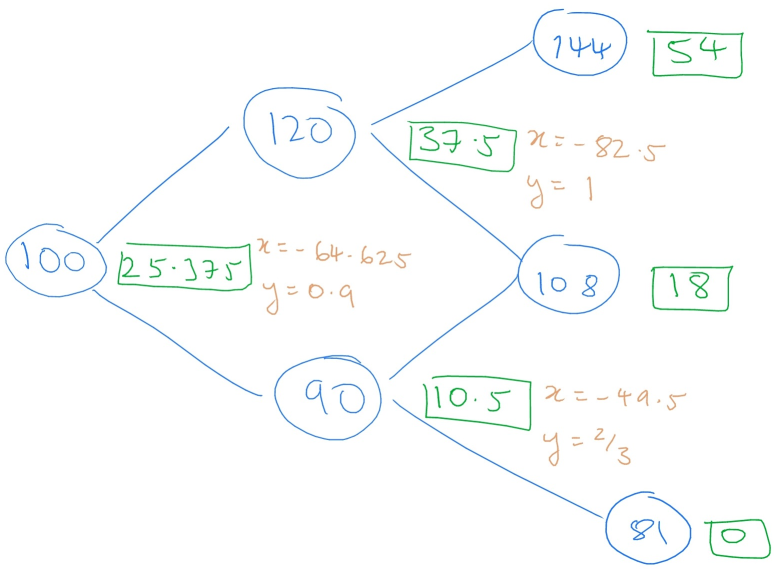
1. Consider the binomial model with $r = \frac{1}{11}$, $d = 0.9$, $u = 1.2$, $s = 100$ and time steps $t = 0, 1, 2$.

   (a) Draw a recombining tree of the stock price process, for time $t = 0, 1, 2$.

   (b) Find the value, at time $t = 0$, of a European call option that gives its holder the option to purchase one unit of stock at time $t = 2$ for a strike price $K = 90$. Write down the hedging strategy that replicates the value of this contract, at all nodes of your tree.

   You may annotate your tree from (a) to answer (b).

   Solution. As in the lecture notes, we write the value of a unit of stock (in blue) inside the nodes of the tree, to answer (a), and write the value of the contingent claim at the various nodes, in square boxes (in green), next to the nodes themselves; the answer to the first part of (b) appears at the root node. For the second part of (b), the replicating portfolios $h = (x, y)$ that would be held at each node are written (in orange) as $x = \ldots, y = \ldots$.



   (To find these numbers you will need to either solve suitable linear equations and/or use the risk neutral valuation formula – see the lecture notes for details.)

   [2A, for (a)]

   [3A + 4M, for (b)]

2. Let $S_n = \sum_{i=1}^n X_i$, be a random walk, in which $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common distribution $\mathbb{P}[X_i = \frac{1}{i^2}] = \mathbb{P}[X_i = -\frac{1}{i^2}] = \frac{1}{2}$.

   (a) Show that $\mathbb{E}[|S_n|] \le \sum\limits_{i=1}^n \frac{1}{i^2}$, and that (S_n) is uniformly bounded in L^1.

   (b) Show there exists a random variable S_∞ such that $S_n \stackrel{a.s.}{\to} S_\infty$ as $n\to\infty$.

   (c) Determine whether (S_n) is bounded in L^2, and briefly state what else (if anything) can be deduced about S_∞ as a consequence.

Solution.

(a) Using the triangle inequality, and monotonicity of $\mathbb{E}$, [1J]

$$\mathbb{E}[|S_n|] = \mathbb{E}\left[\left|\sum_{i=1}^n X_i\right|\right] \le \mathbb{E}\left[\sum_{i=1}^n |X_i|\right] = \mathbb{E}\left[\sum_{i=1}^n \frac{1}{i^2}\right] = \sum_{i=1}^n \frac{1}{i^2}.$$

[1A] From the above we have $\mathbb{E}[|S_n|] \le \sum_{i=1}^\infty \frac{1}{i^2}$, the right hand side of which is finite and independent of n. [1J] Hence $\sup_n \mathbb{E}[|S_n|] < \infty$. [1A]

Pitfall: The definition of the sequence (S_n) being uniformly bounded in L^1 is that $\sup_n \mathbb{E}[|S_n|] < \infty$. This is different to requiring that the random variable S_n is in L^1 for all n, which requires only that $\mathbb{E}[|S_n|] < \infty$ for all $n \in \mathbb{N}$.

(b) We aim to use the martingale convergence theorem.[1M] We must check that (S_n) is a martingale.

We use the filtration $\mathcal{F}_n = \sigma(X_i \;:\; i = 1,\dots,n)$. Since $X_i \in m\mathcal{F}_n$, we have $S_n \in m\mathcal{F}_n$.[1J] We have already shown in (a) that $\mathbb{E}[|S_n|] < \infty$, so $S_n \in L^1$. [1J] Lastly,

$$\begin{aligned}\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[X_{n+1} + S_n \mid \mathcal{F}_n]\\ &= \mathbb{E}[X_{n+1}] + S_n\\ &= S_n.\end{aligned}$$

[1A] Here we use that $S_n \in m\mathcal{F}_n$, [1J] that X_{n+1} is independent of $\mathcal{F}_n$, [1J] and that $\mathbb{E}[X_{n+1}] = 0$.

(c) Note that for $i \ne j$ we have $\mathbb{E}[X_iX_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] = 0$, by independence. [1J] We have

$$\begin{aligned}\mathbb{E}[|S_n|^2] = \mathbb{E}[S_n^2] = \mathbb{E}\left[\sum_{i=1}^n\sum_{j=1}^n X_iX_j\right] &= \sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[X_iX_j\right]\\ = \sum_{i=1}^n\left(\mathbb{E}[X_i^2] + \sum_{\substack{j=1\\ j\ne i}}^n \mathbb{E}[X_iX_j]\right) &= \sum_{i=1}^n\left(\frac{1}{i^4} + \sum_{\substack{j=1\\ j\ne i}}^n 0\right) = \sum_{i=1}^n \frac{1}{i^4}.\end{aligned}$$

[2A]

Therefore, for any n we have $\mathbb{E}[S_n^2] \le \sum_{i=1}^\infty i^{-4}$. Since the right hand side is finite and independent of n we have that $\sup_n \mathbb{E}[S_n^2] < \infty$, hence (S_n) is bounded in L^2. [1J]

Hence, the second version of the martingale convergence theorem applies, which gives us that $\mathbb{E}[S_n] \to \mathbb{E}[S_\infty]$ and $\mathrm{Var}(S_n) \to \mathrm{Var}(S_\infty)$. [1M]

Since $\mathbb{E}[S_n] = 0$ this means that $\mathbb{E}[S_\infty] = 0$ and also $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2]$, which in turn means that $\mathrm{Var}(S_\infty) = \sum_{i=1}^\infty i^{-4}$. [1J]

Total marks: 25