MASx52: Assignment 4

1. Let B_t be a standard Brownian motion.

   (a) Write down the distribution of B_t, and write down $\mathbb{E}[B_t]$ and $\mathbb{E}[B_t^2]$.

   (b) Let $0 \leq u \leq t$. Show that $\mathbb{E}[(B_t - B_u)^2 \,|\, \mathcal{F}_u] = t - u$.

2. Write down the following stochastic differential equations in integral form, over the time interval $[0, t]$.

   (a) $dX_t = 2(X_t + 1)\,dt + 2B_t\,dB_t$.

   (b) $dY_t = 3Y_t\,dt$.

   Write down a differential equation satisfied by Y_t, and find its solution with the initial condition $Y_0 = 1$.

   Suppose that $X_0 = 1$. Show that $f(t) = \mathbb{E}[X_t]$ satisfies $f'(t) = 2f(t) + 2$ and hence find $f(t)$.

3. Use Ito's formula to calculate the stochastic differential of dZ_t where

   (a) $Z_t = tB_t$

   (b) $Z_t = 1 + t^2 X_t$ where $dX_t = \mu\,dt + \sigma B_t\,dB_t$ and μ, σ are deterministic constants.

   (c) $Z_t = e^{-2t} S_t$ where $dS_t = 2S_t\,dt + 5S_t\,dB_t$.

   In which cases is Z_t is a martingale?

4. Let S_t be a geometric Brownian motion, with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and (deterministic) initial condition S_0.

   (a) Find $\mathbb{E}[S_t]$ and deduce that S_t is not a Brownian motion when $\mu \neq 0$.

   (b) Is S_t a Brownian motion when $\mu = 0$?